Introduction to Programming (in C++)

Sorting

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Sorting

- Let elem be a type with a ≤ operation, which is a total order

- A vector<elem> v is (increasingly) sorted if

  for all i with $0 \leq i < v.size()-1$, $v[i] \leq v[i+1]$

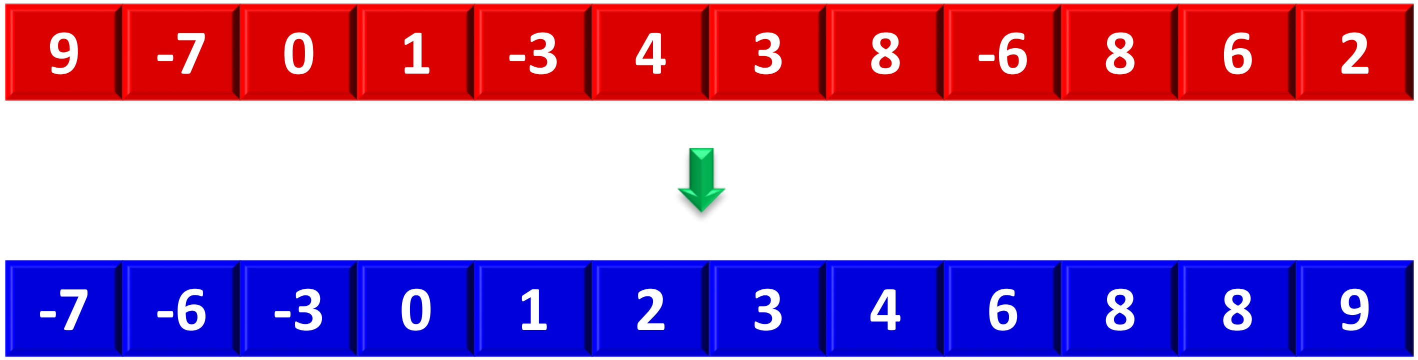
- Equivalently:

  if $i < j$ then $v[i] \leq v[j]$

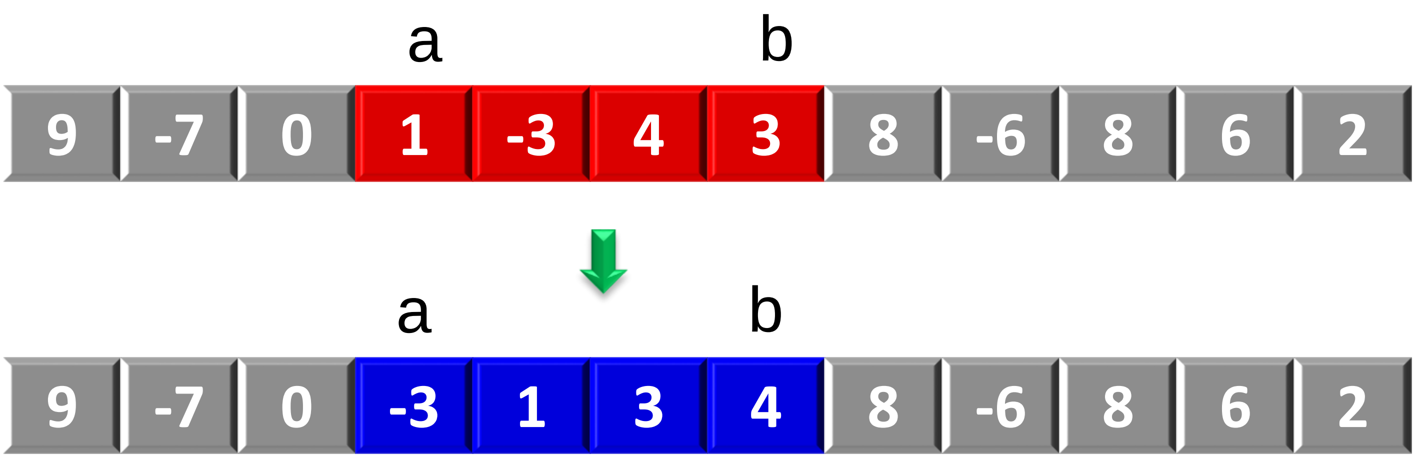
- A fundamental, very common problem: **sort v**

  Order the elements in v and leave the result in v

Sorting



- Another common task: **sort v[a..b]**



Sorting

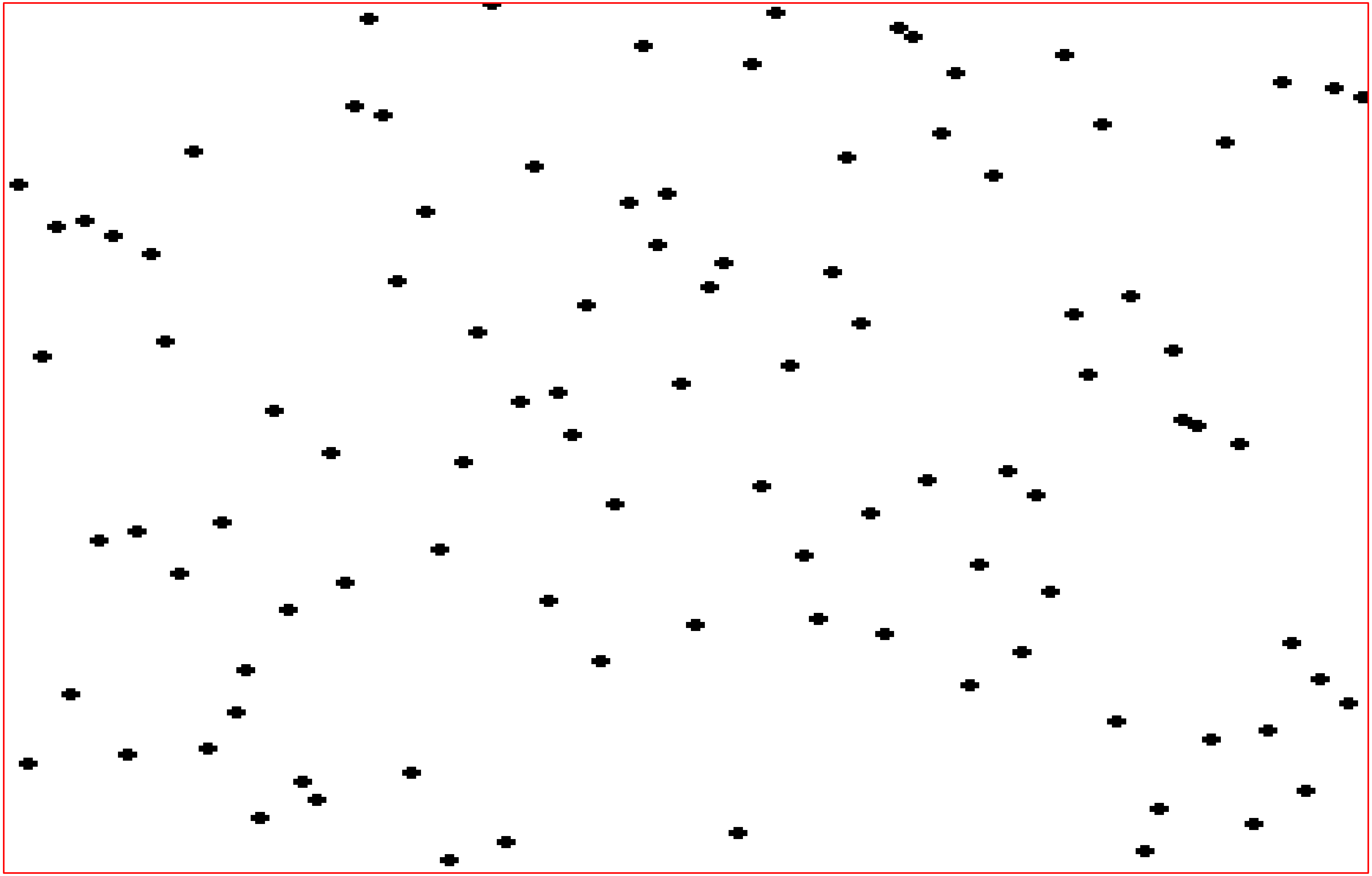
- We will look at four sorting algorithms:
  - Selection Sort
  - Insertion Sort
  - Bubble Sort
  - Merge Sort

- Let us consider a vector v of n elems (n = v.size())
  - Insertion, Selection and Bubble Sort make a number of operations on elems proportional to n^2
  - Merge Sort is proportional to $n \cdot \log_2 n$: faster except for very small vectors

Selection Sort

- Observation: in the sorted vector, v[0] is the smallest element in v
- The second smallest element in v must go to v[1]…
- … and so on
- At the i-th iteration, select the i-th smallest element and place it in v[i]

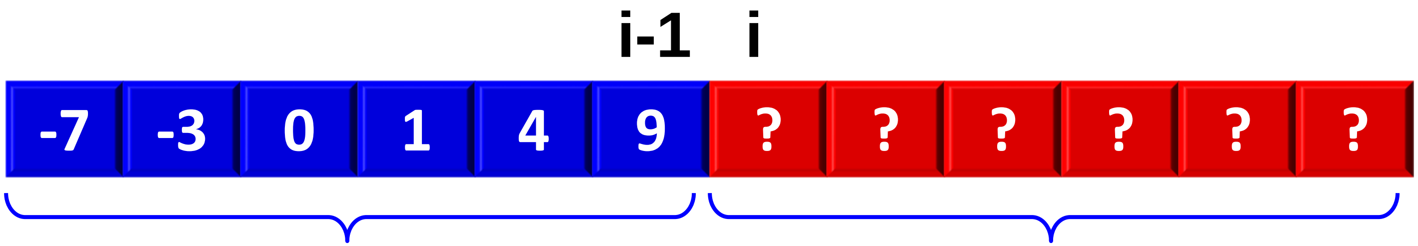
Selection Sort



From http://en.wikipedia.org/wiki/Selection_sort

Selection Sort

- Selection sort keeps this invariant:



this is sorted
and contains the i-1 smallest elements

this may not be sorted…
but all elements here are larger than or equal to the elements in the sorted part

Selection Sort

```
// Pre:  --
// Post: v is now increasingly sorted

void selection_sort(vector<elem>& v) {
    int last = v.size() - 1;
    for (int i = 0; i < last; ++i) {
        int k = pos_min(v, i, last);
        swap(v[k], v[i]);
    }
}
```

```
// Invariant: v[0..i-1] is sorted and
//            if a < i <= b then v[a] <= v[b]
```

Note: when i=v.size()-1, v[i] is necessarily the largest element. Nothing to do.

Selection Sort

```
// Pre: 0 <= left <= right < v.size()
// Returns pos such that left <= pos <= right
// and v[pos] is smallest in v[left..right]

int pos_min(const vector<elem>& v, int left, int right) {
    int pos = left;
    for (int i = left + 1; i <= right; ++i) {
        if (v[i] < v[pos]) pos = i;
    }
    return pos;
}
```

Selection Sort

- At the i-th iteration, Selection Sort makes
  - up to v.size()-1-i comparisons among elems
  - 1 swap (=3 elem assignments) per iteration

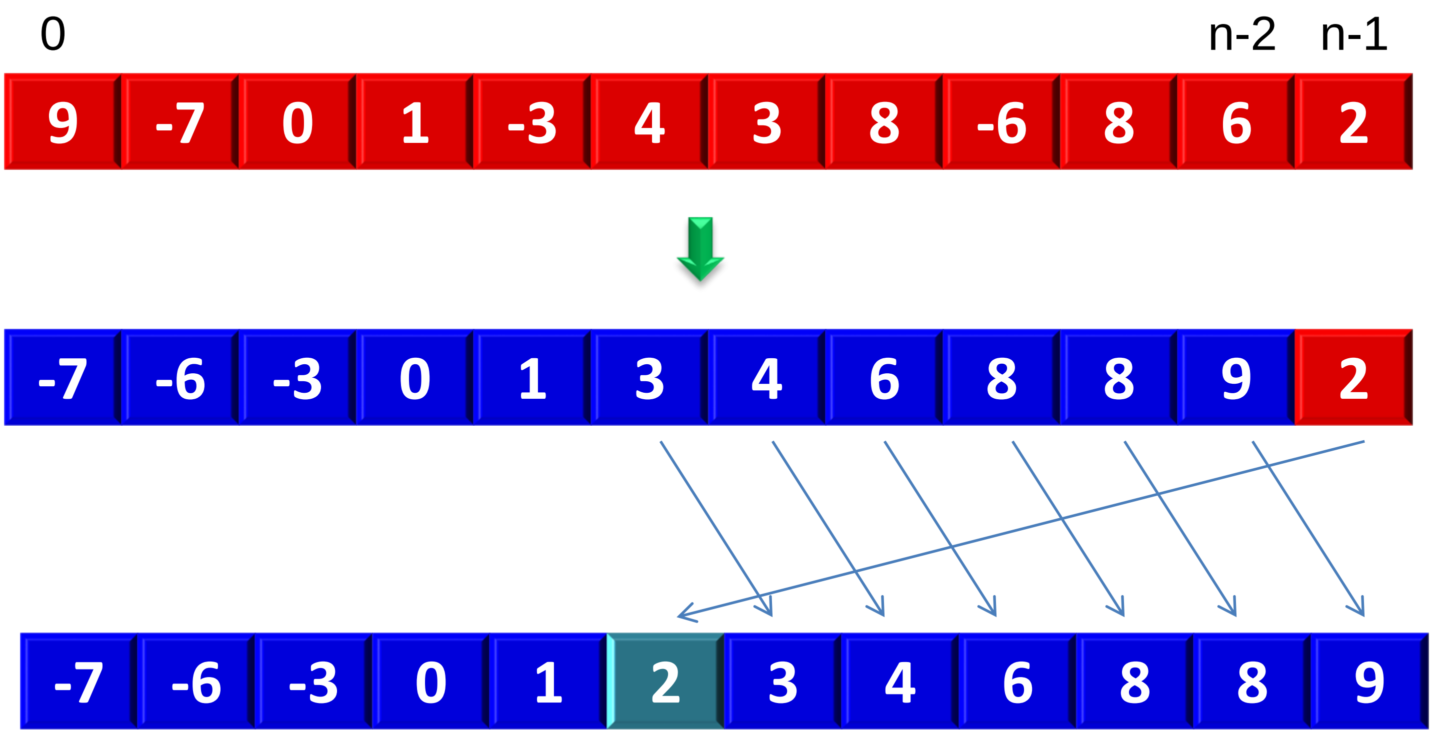
- The total number of comparisons for a vector of size n is:

$$(n-1)+(n-2)+\ldots+1 = n(n-1)/2 \approx n^2/2$$

- The total number of assignments is 3(n-1).

Insertion Sort

- Let us use induction:
  - If we know how to sort arrays of size n-1,
  - do we know how to sort arrays of size n?



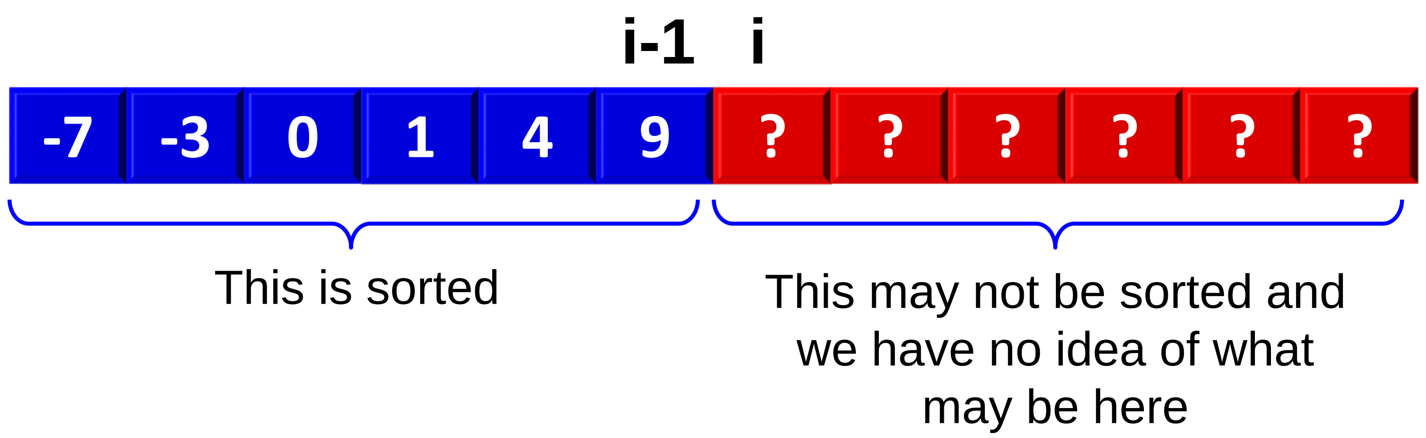
Insertion Sort

- Insert x=v[n-1] in the right place in v[0..n-1]

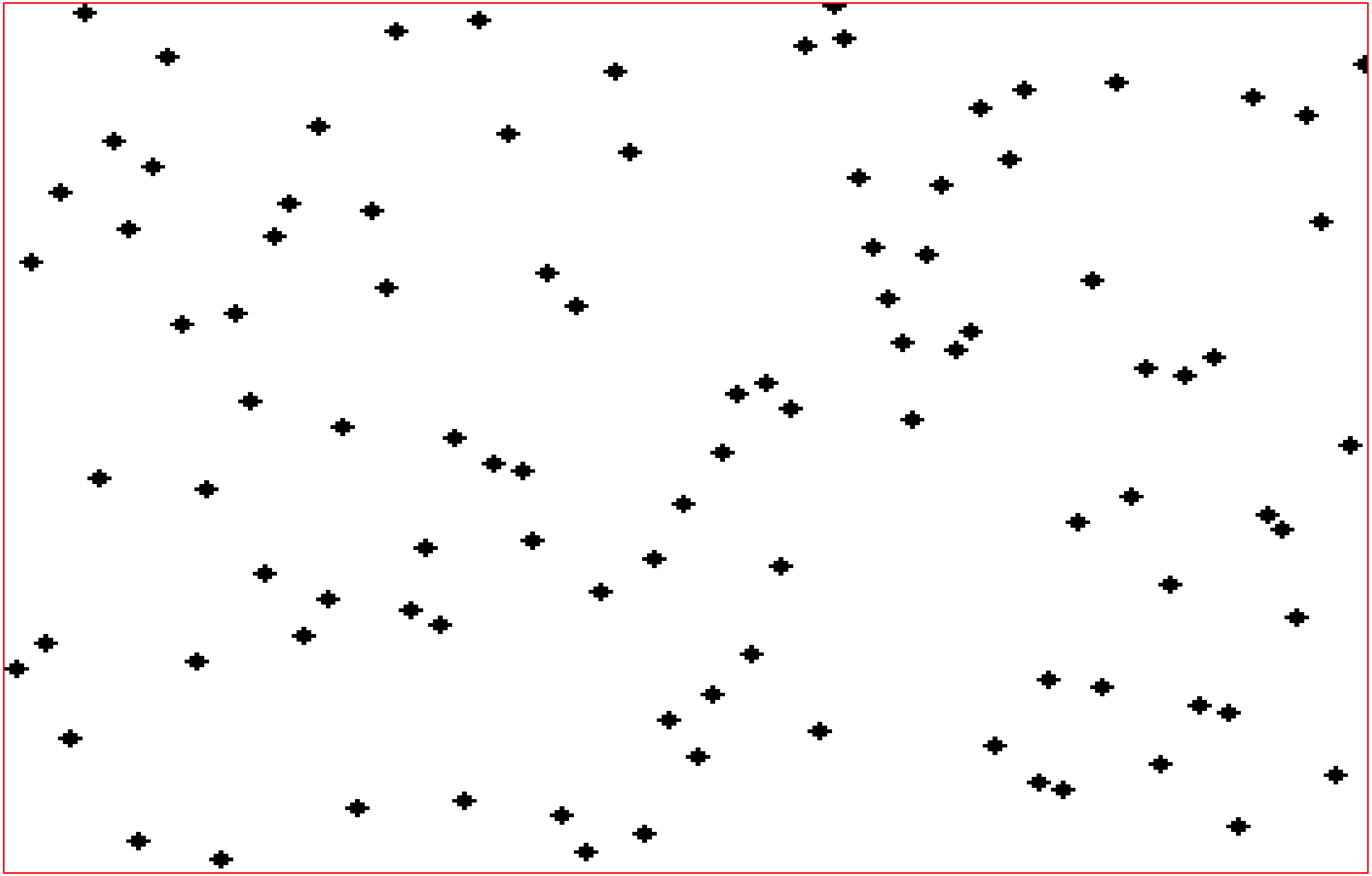
- Two ways:
  - Find the right place, then shift the elements
  - Shift the elements to the right until one ≤ x is found

Insertion Sort

- Insertion sort keeps this invariant:



Insertion Sort



From http://en.wikipedia.org/wiki/Insertion_sort

Insertion Sort

```
// Pre:  --
// Post: v is now increasingly sorted
void insertion_sort(vector<elem>& v) {
    for (int i = 1; i < v.size(); ++i) {
        elem x = v[i];
        int j = i;
        while (j > 0 and v[j - 1] > x) {
            v[j] = v[j - 1];
            --j;
        }
        v[j] = x;
    }
}
```

```
// Invariant: v[0..i-1] is sorted in ascending order
```

Insertion Sort

- At the i-th iteration, Insertion Sort makes up to i comparisons and up to i+2 assignments of type elem

- The total number of comparisons for a vector of size n is, at most:

$$1 + 2 + \ldots + (n-1) = n(n-1)/2 \approx n^2/2$$

- At the most, $n^2/2$ assignments

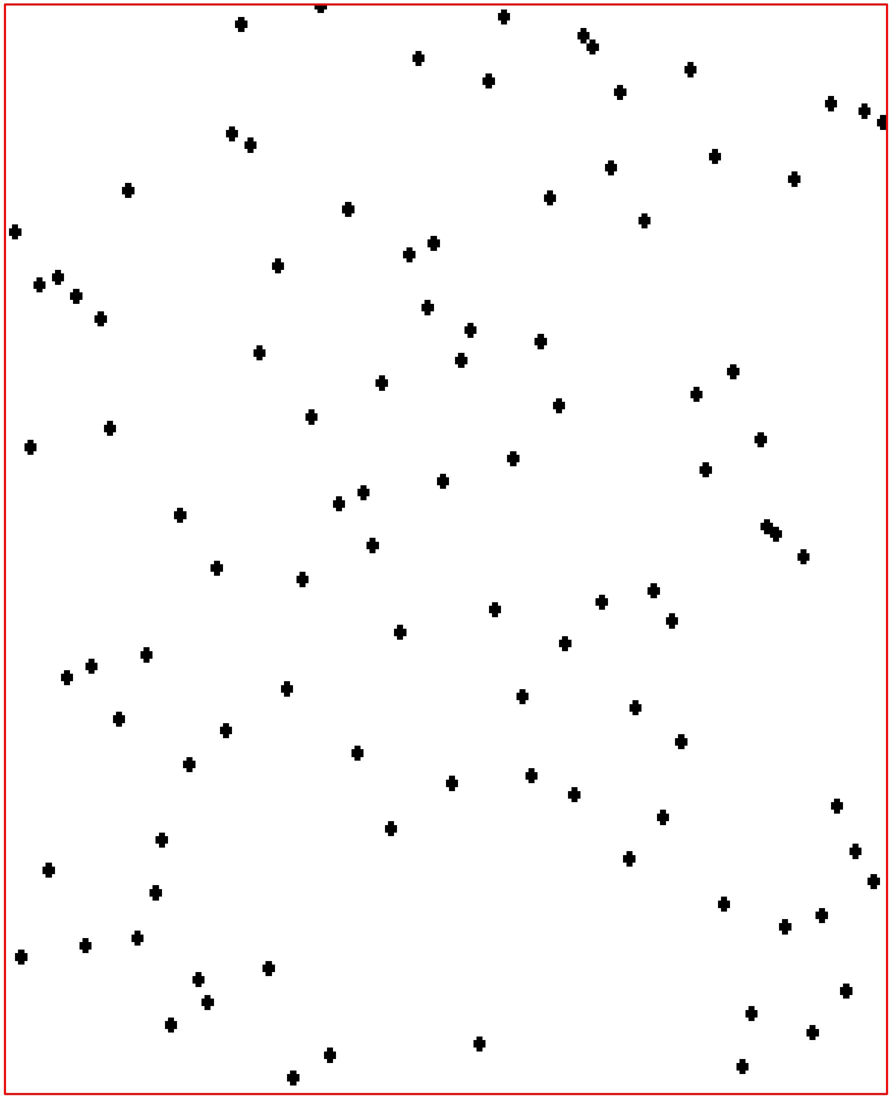
- But about $n^2/4$ in typical cases

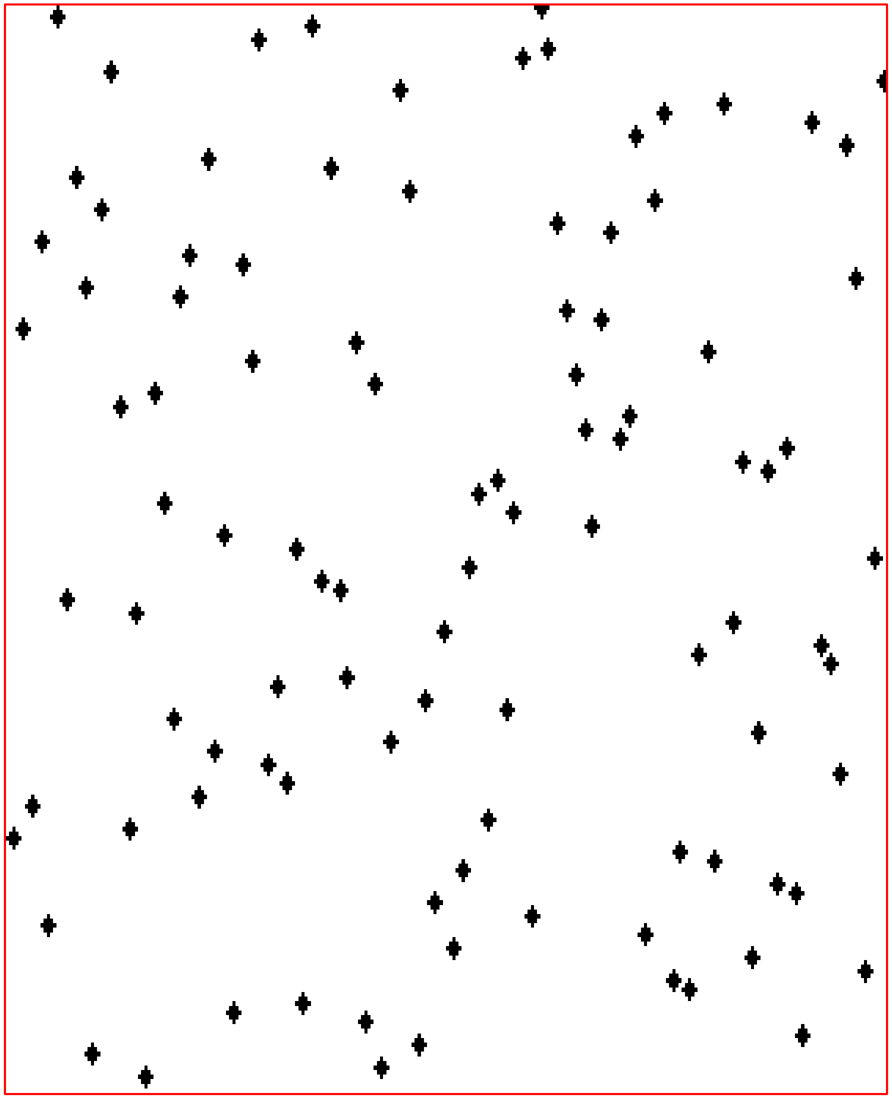
Selection Sort vs. Insertion Sort

2	-1	5	0	-3	9	4
-3	-1	5	0	2	9	4
-3	-1	5	0	2	9	4
-3	-1	0	5	2	9	4
-3	-1	0	2	5	9	4
-3	-1	0	2	4	9	5
-3	-1	0	2	4	5	9

2	-1	5	0	-3	9	4
-1	2	5	0	-3	9	4
-1	2	5	0	-3	9	4
-1	0	2	5	-3	9	4
-3	-1	0	2	5	9	4
-3	-1	0	2	5	9	4
-3	-1	0	2	4	5	9

Selection Sort vs. Insertion Sort





Evaluation of complex conditions

```
void insertion_sort(vector<elem>& v) {
    for (int i = 1; i < v.size(); ++i) {
        elem x = v[i];
        int j = i;
        while (j > 0 and v[j - 1] > x) {
           v[j] = v[j - 1];
           --j;
        }
        v[j] = x;
    }
}
```

- How about: `while (v[j – 1] > x and j > 0)` ?
- Consider the case for `j = 0` → evaluation of `v[-1]` (error !)
- How are complex conditions really evaluated?

Evaluation of complex conditions

- Many languages (C, C++, Java, PHP, Python) use the *short-circuit evaluation* (also called *minimal* or *lazy* evaluation) for Boolean operators.

- For the evaluation of the Boolean expression

  expr1 ***op*** expr2

  expr2 is only evaluated if *expr1* does not suffice to determine the value of the expression.

- Example: `(j > 0 and v[j-1] > x)`

  `v[j-1]` is only evaluated when `j>0`

Evaluation of complex conditions

- In the following examples:

```
n != 0 and sum/n > avg

n == 0 or  sum/n > avg
```

  sum/n will never execute a division by zero.

- Not all languages have short-circuit evaluation. Some of them have *eager evaluation* (all the operands are evaluated) and some of them have both.

- The previous examples could potentially generate a runtime error (division by zero) when eager evaluation is used.

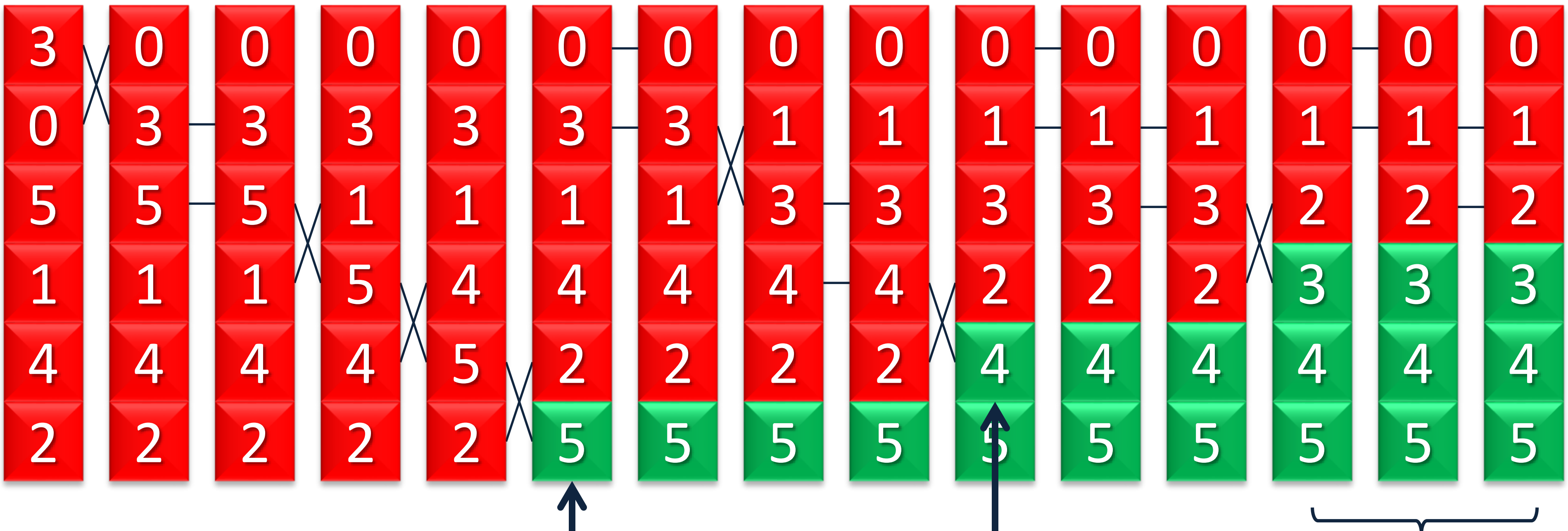
- Tip: short-circuit evaluation helps us to write more efficient programs, but cannot be used in all programming languages.

Bubble Sort

- A simple idea: traverse the vector many times, swapping adjacent elements when they are in the wrong order.

- The algorithm terminates when no changes occur in one of the traversals.

Bubble Sort

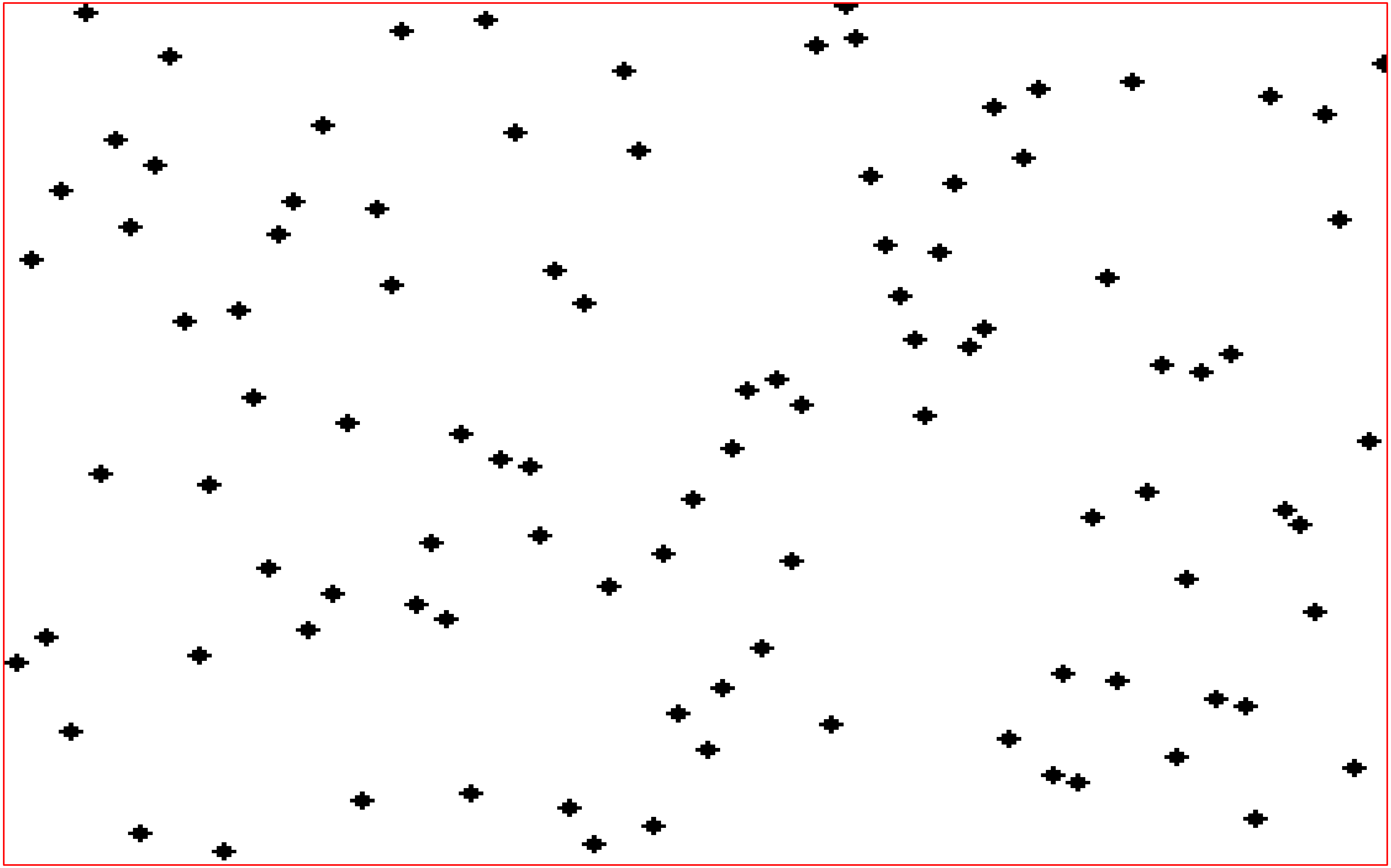


The largest element is well-positioned after the first iteration.

The second largest element is well-positioned after the second iteration.

The vector is sorted when no changes occur during one of the iterations.

Bubble Sort



From http://en.wikipedia.org/wiki/Bubble_sort

Bubble Sort

```
void bubble_sort(vector<elem>& v) {
    bool sorted = false;
    int last = v.size() - 1;
    while (not sorted) { // Stop when no changes
        sorted = true;
        for (int i = 0; i < last; ++i) {
            if (v[i] > v[i + 1]) {
                swap(v[i], v[i + 1]);
                sorted = false;
            }
        }
        // The largest element falls to the bottom
        --last;
    }
}
```

Observation: at each pass of the algorithm, all elements after the last swap are sorted.

Bubble Sort

```
void bubble_sort(vector<elem>& v) {
    int last = v.size() - 1;
    while (last > 0) {
        int last_swap = 0; // Last swap at each iteration
        for (int i = 0; i < last; ++i) {
            if (v[i] > v[i + 1]) {
                swap(v[i], v[i + 1]);
                last_swap = i;
            }
        }
        last = last_swap; // Skip the sorted tail
    }
}
```

Bubble Sort

- Worst-case analysis:
  - The first pass makes n-1 swaps
  - The second pass makes n-2 swaps
  - …
  - The last pass makes 1 swap

- The worst number of swaps:

$$1 + 2 + \ldots + (n-1) = n(n-1)/2 \approx n^2/2$$

- It may be efficient for nearly-sorted vectors.

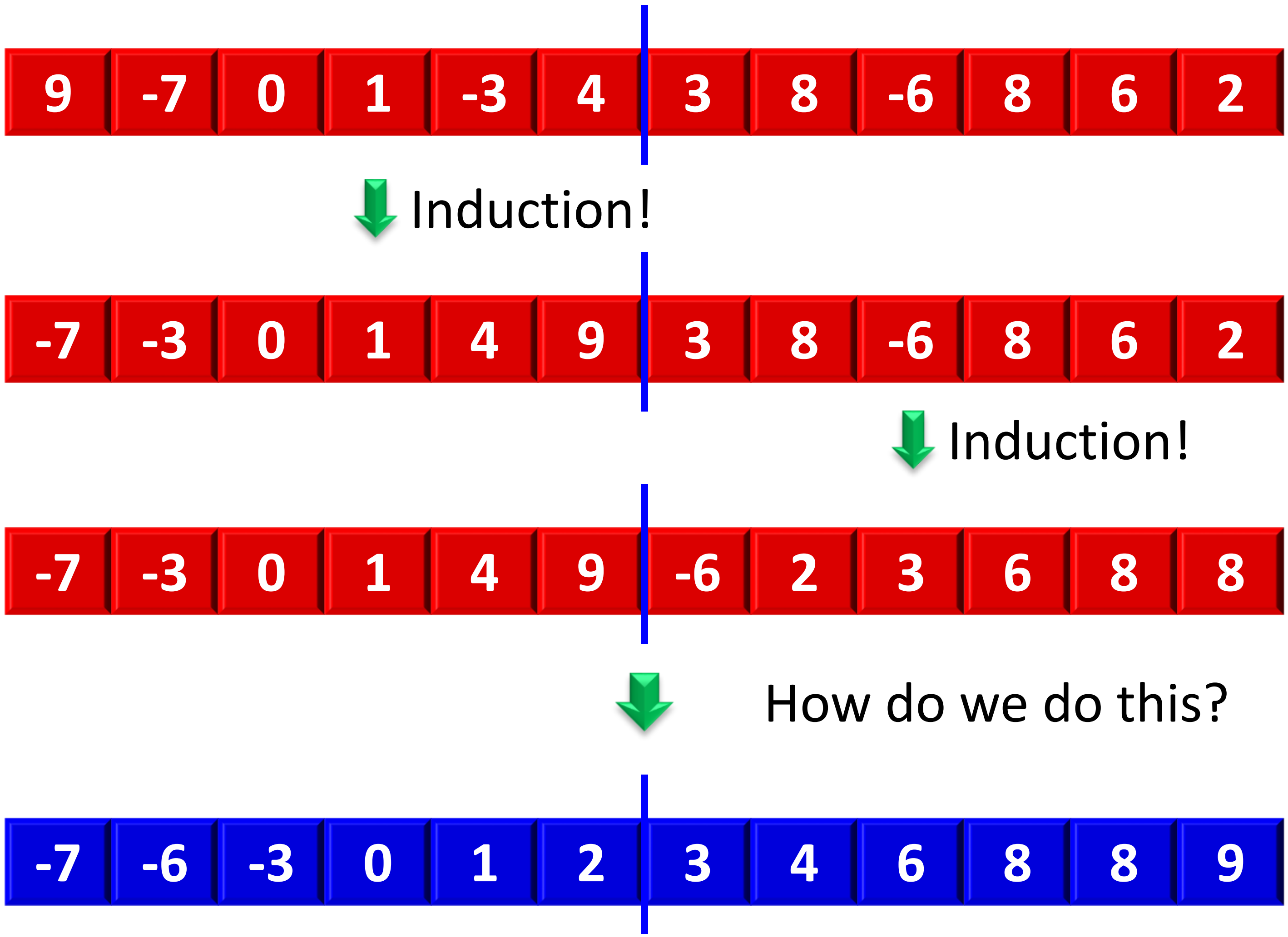
- In general, bubble sort is one of the least efficient algorithms. It is not practical when the vector is large.

Merge Sort

- Recall our induction for Insertion Sort:
  - suppose we can sort vectors of size **n-1**,
  - can we now sort vectors of size **n**?

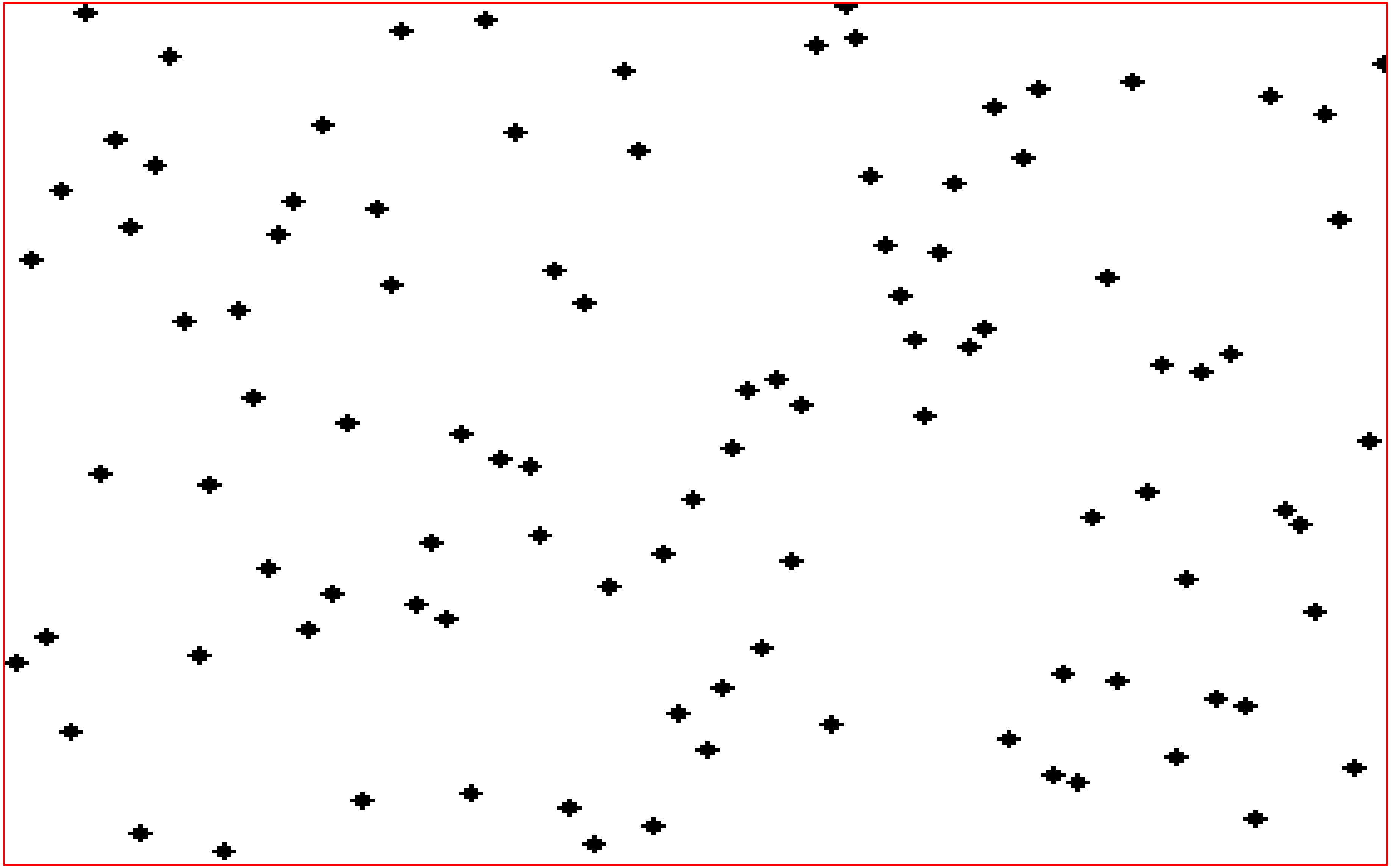
- What about the following:
  - suppose we can sort vectors of size **n/2**,
  - can we now sort vectors of size **n**?

Merge Sort



9 -7 0 1 -3 4 3 8 -6 8 6 2
Induction!
-7 -3 0 1 4 9 3 8 -6 8 6 2
Induction!
-7 -3 0 1 4 9 -6 2 3 6 8 8
How do we do this?
-7 -6 -3 0 1 2 3 4 6 8 8 9

Merge Sort



From http://en.wikipedia.org/wiki/Merge_sort

Merge Sort

- We have seen almost what we need!

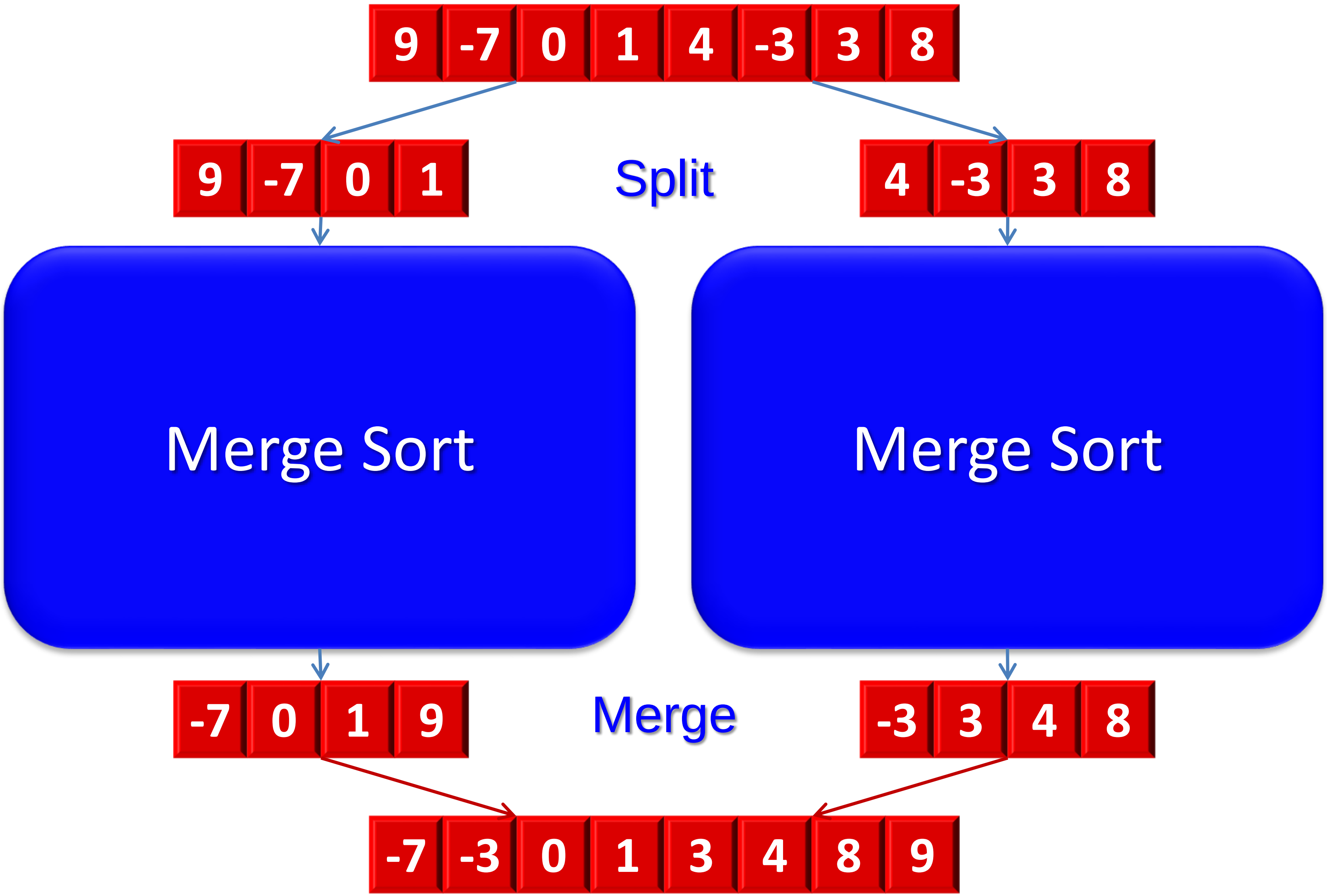
```
// Pre: A and B are sorted in ascending order
// Returns the sorted fusion of A and B

vector<elem> merge(const vector<elem>& A,
                   const vector<elem>& B);
```

- Now, **v[0..n/2-1]** and **v[n/2..n-1]** are sorted in ascending order.

- Merge them into an auxiliary vector of size n, then copy back to v.

Merge Sort



9 -7 0 1 4 -3 3 8
9 -7 0 1
Split
4 -3 3 8
Merge Sort
Merge Sort
-7 0 1 9
Merge
-3 3 4 8
-7 -3 0 1 3 4 8 9

Merge Sort

```
// Pre:  0 <= left <= right < v.size()
// Post: v[left..right] has been sorted increasingly

void merge_sort(vector<elem>& v, int left, int right) {
    if (left < right) {
        int m = (left + right)/2;
        merge_sort(v, left, m);
        merge_sort(v, m + 1, right);
        merge(v, left, m, right);
    }
}
```

Merge Sort – merge procedure

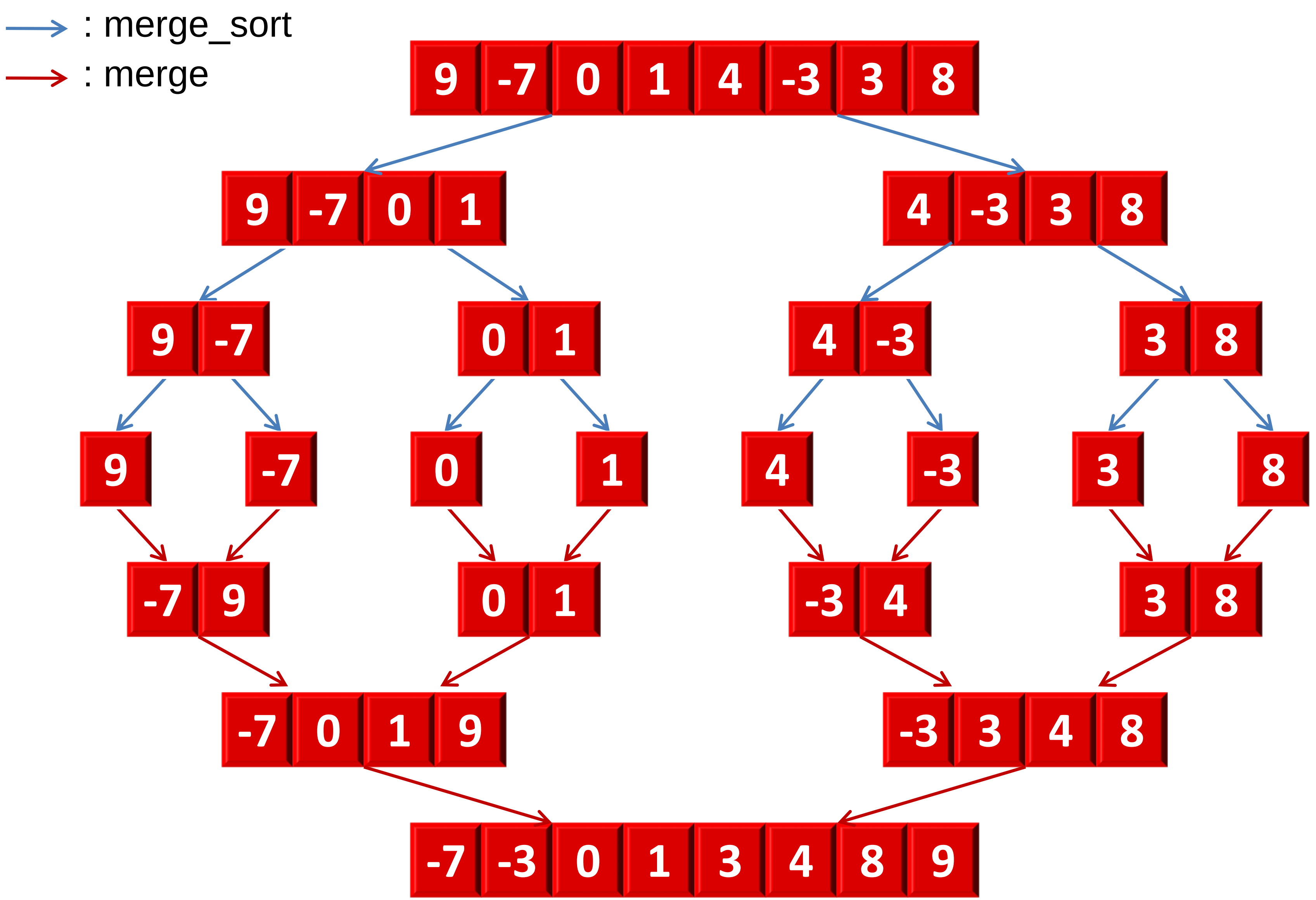
```
// Pre:  0 <= left <= mid < right < v.size(), and
//       v[left..mid], v[mid+1..right] are both sorted increasingly
// Post: v[left..right] is now sorted

void merge(vector<elem>& v, int left, int mid, int right) {
    int n = right - left + 1;
    vector<elem> aux(n);
    int i = left;
    int j = mid + 1;
    int k = 0;
    while (i <= mid and j <= right) {
        if (v[i] <= v[j]) { aux[k] = v[i]; ++i; }
        else { aux[k] = v[j]; ++j; }
        ++k;
    }

    while (i <= mid) { aux[k] = v[i]; ++k; ++i; }
    while (j <= right) { aux[k] = v[j]; ++k; ++j; }

    for (k = 0; k < n; ++k) v[left+k] = aux[k];
}
```

Merge Sort



: merge_sort
: merge
9 -7 0 1 4 -3 3 8
9 -7 0 1
4 -3 3 8
9 -7
0 1
4 -3
3 8
9
-7
0
1
4
-3
3
8
-7 9
0 1
-3 4
3 8
-7 0 1 9
-3 3 4 8
-7 -3 0 1 3 4 8 9

Merge Sort

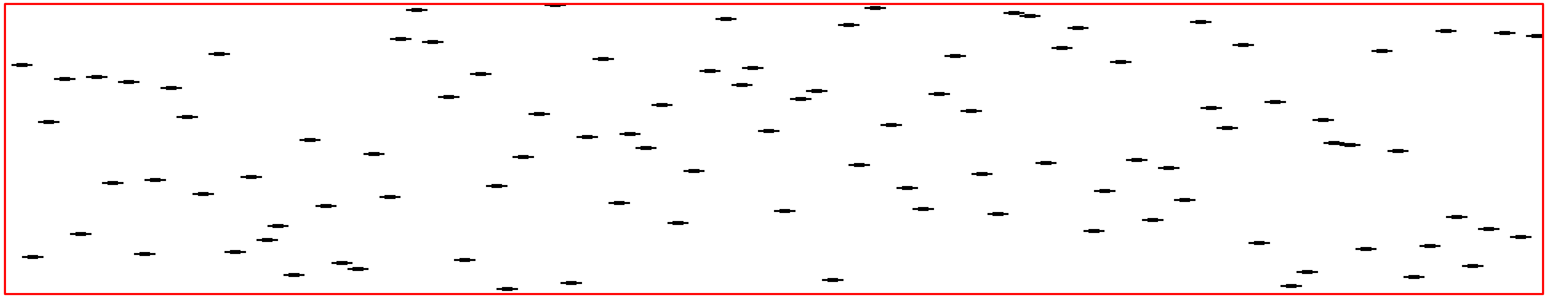
- How many elem comparisons does Merge Sort do?
  - Say v.size() is n, a power of 2
  - merge(v,L,M,R) makes k comparisons if k=R-L+1
  - We call merge $\frac{n}{2^i}$ times with R-L=2^i
  - The total number of comparisons is

$$\sum_{i=1}^{\log_2 n} \frac{n}{2^i} \cdot 2^i = n \cdot \log_2 n$$

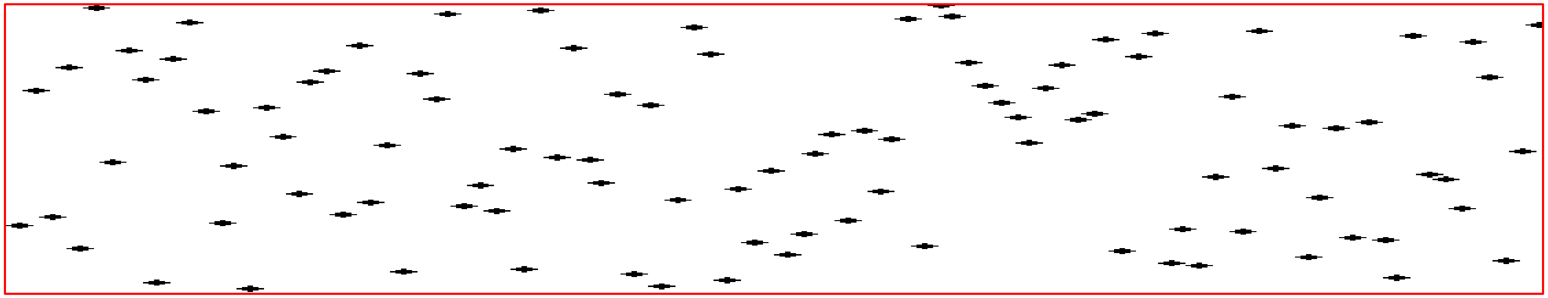
The total number of elem assignments is $2n \cdot \log_2 n$

Comparison of sorting algorithms

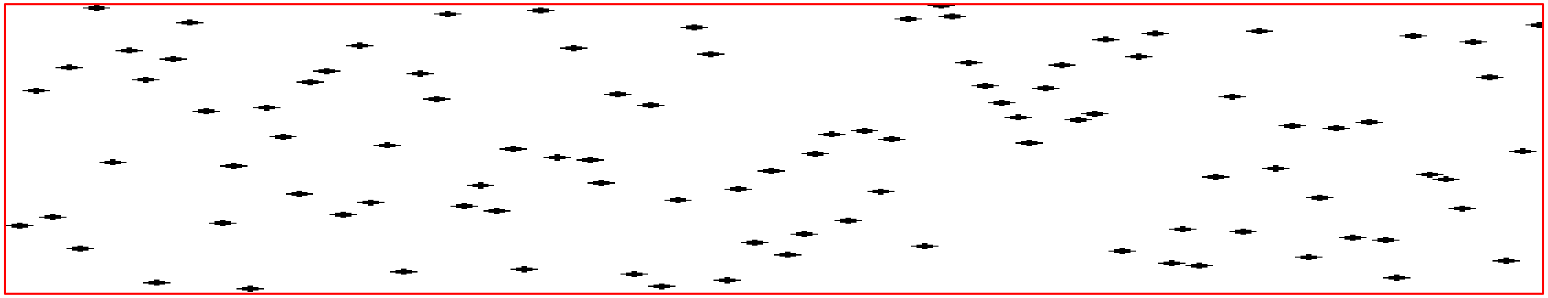
Selection



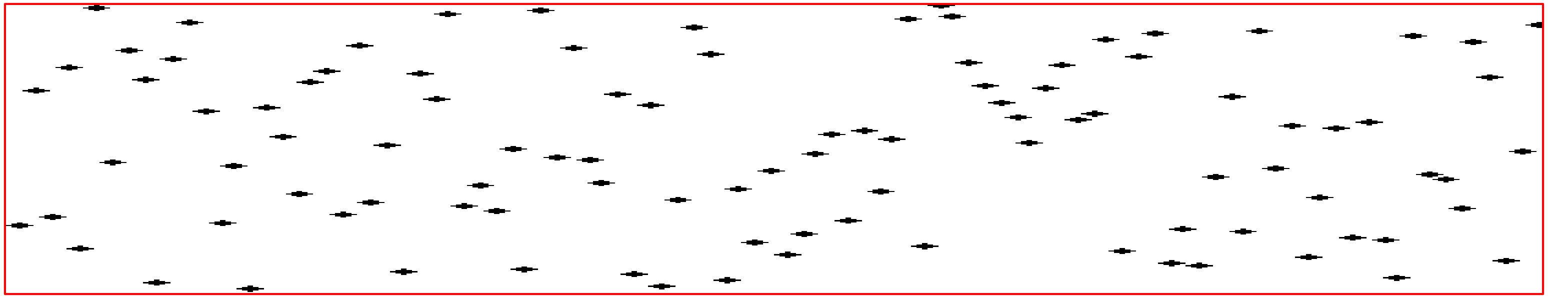
Insertion



Bubble



Merge



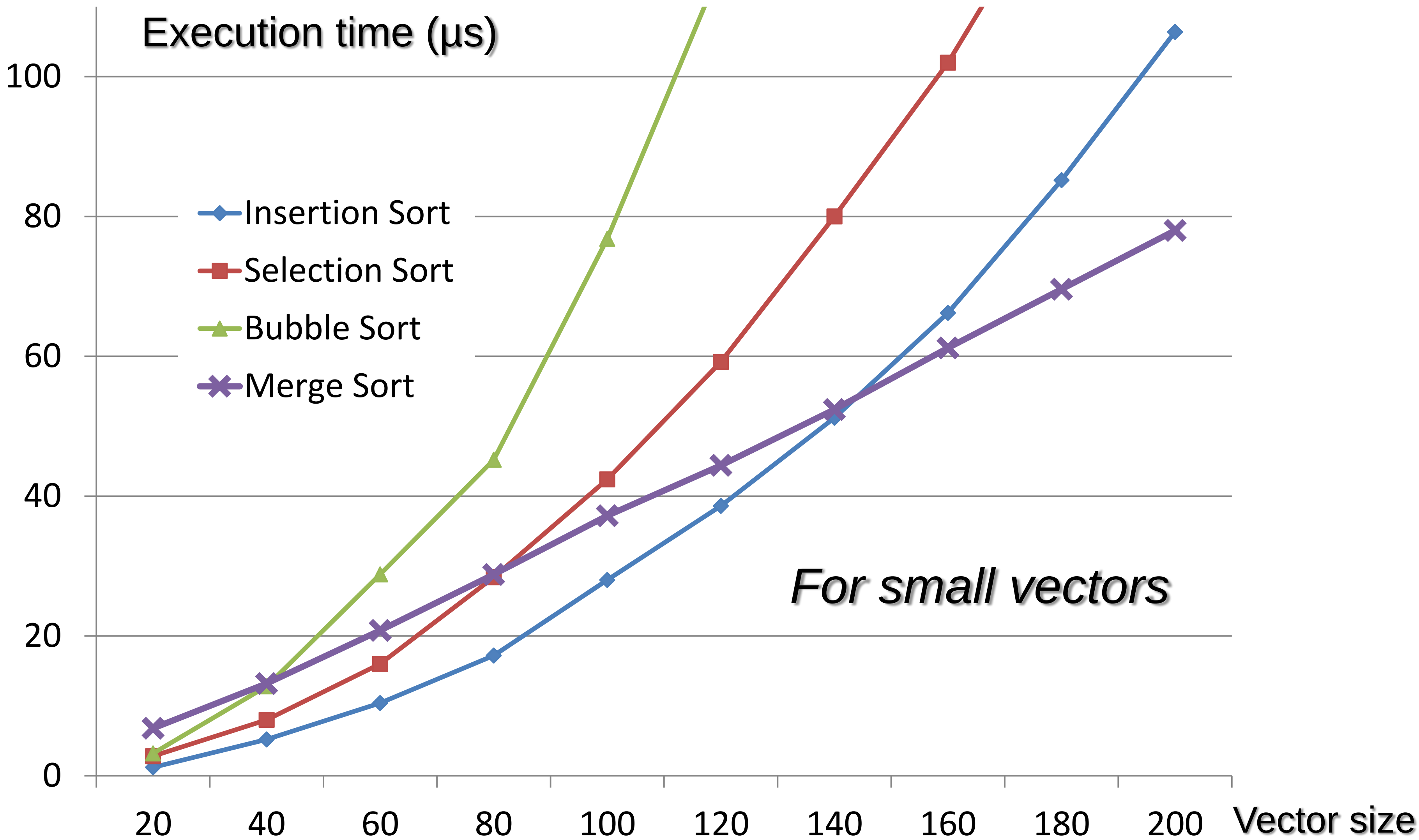
Comparison of sorting algorithms

- Approximate number of comparisons:

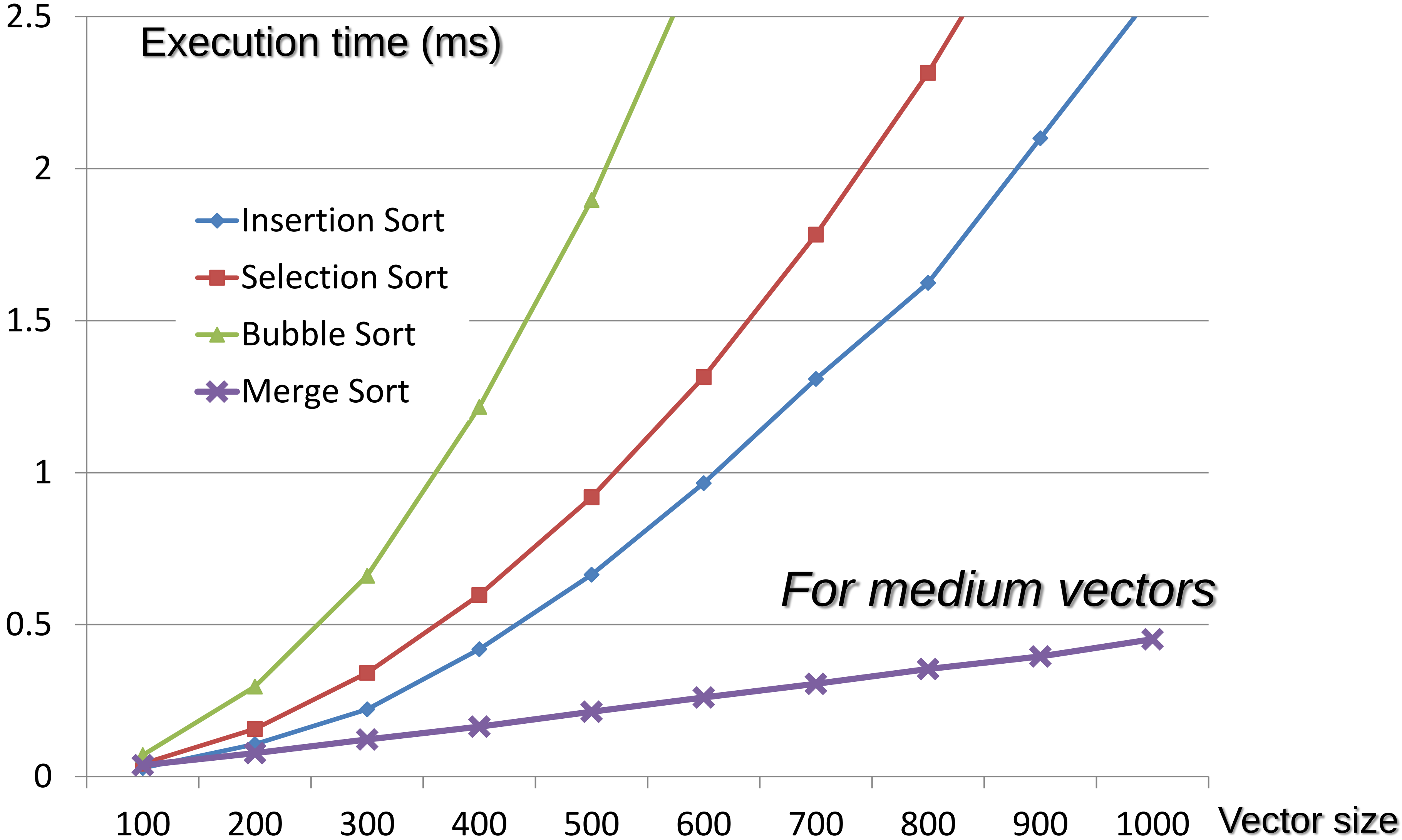
n = v.size()	10	100	1,000	10,000	100,000
Insertion, Selection and Bubble Sort ($\approx n^2/2$)	50	5,000	500,000	50,000,000	5,000,000,000
Merge Sort ($\approx n \cdot log_2 n$)	67	1,350	20,000	266,000	3,322,000

- **Note:** it is known that every general sorting algorithm must do at least $n \cdot log_2 n$ comparisons.

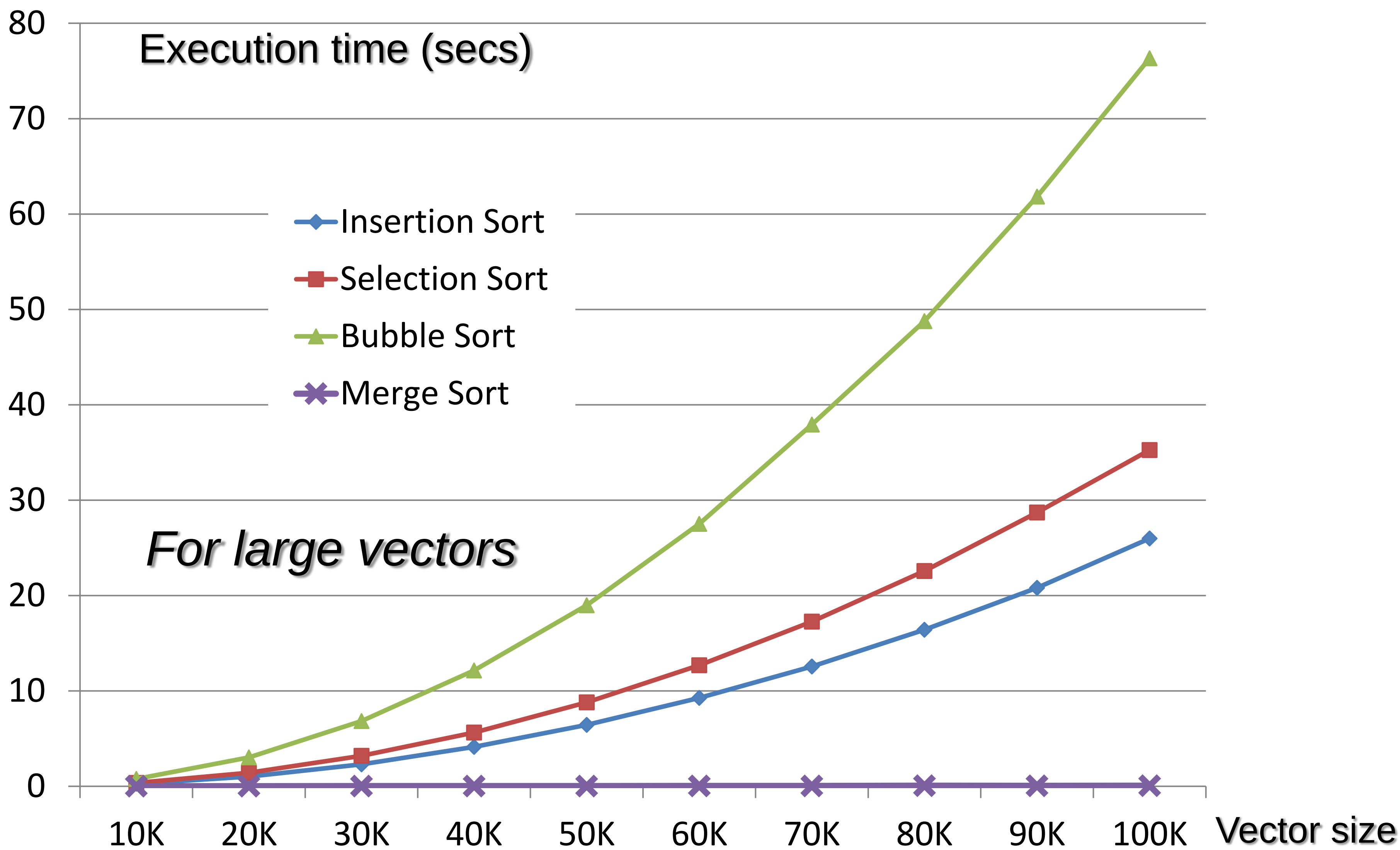
Comparison of sorting algorithms



Comparison of sorting algorithms



Comparison of sorting algorithms



Other sorting algorithms

- There are many other sorting algorithms.
- The most efficient algorithm for general sorting is *quick sort* (C.A.R. Hoare).
  - The worst case is proportional to n^2
  - The average case is proportional to $n \cdot \log_2 n$, but it usually runs faster than all the other algorithms
  - It does not use any auxiliary vectors
- Quick sort will not be covered in this course.

Sorting with the C++ library

- A sorting procedure is available in the C++ library
- It probably uses a quicksort algorithm
- To use it, include:

```
#include <algorithm>
```

- To increasingly sort a vector v (of int’s, double’s, string’s, etc.), call:

```
sort(v.begin(), v.end());
```

Sorting with the C++ library

- To sort with a different comparison criteria, call

```
sort(v.begin(), v.end(), comp);
```

- For example, to sort int's <u>decreasingly</u>, define:

```
bool comp(int a, int b) {
    return a > b;
}
```

- To sort people by age, then by name:

```
bool comp(const Person& a, const Person& b) {
   if (a.age == b.age) return a.name < b.name;
   else return a.age < b.age;
}
```

Sorting is not always a good idea…

- **Example:** to find the min value of a vector

```
min = v[0];
for (int i=1; i < v.size(); ++i)
    if (v[i] < min) min = v[i];
```

(1)

```
sort(v);
min = v[0];
```

(2)

- **Efficiency analysis:**
  - **Option (1):** n iterations (visit all elements).
  - **Option (2):** $2n \cdot \log_2 n$ moves with a good sorting algorithm (e.g., merge sort)